An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 37 (1999) 31–39

Preinvex Fuzzy Mappings

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(Received May 1997; accepted August 1998)

Abstract—In this paper, we introduce the concepts of preinvex fuzzy mappings, quasi-preinvex fuzzy mappings, and invex sets by modifying the concepts proposed by Noor. We also give two characterization theorems for preinvex fuzzy mappings, and present some new results for preinvex fuzzy mappings, strictly preinvex fuzzy mappings, and quasi-preinvex fuzzy mappings. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Preinvex fuzzy mappings, Invexity, Fuzzy numbers.

1. INTRODUCTION

The concept of fuzzy convexity is important for quantitative and qualitative studies in fuzzy optimization. Noor [1] proposed the notions of preinvex fuzzy mappings and invex sets. These preinvex fuzzy mappings and invex sets are more general than convex fuzzy mappings (see [2]) and convex sets, respectively. Let $\mathcal{F}_0$ denote the set of all fuzzy numbers. Let H be a normed vector space over R^1 and K an invex set with respect to a mapping $\eta : K \times K \to H$. Noor [1] proved that any local minimum of a preinvex fuzzy mapping $F : K \to \mathcal{F}_0$ is a global minimum of F on K, and that a fuzzy mapping $F : K \to \mathcal{F}_0$ is preinvex if and only if its epigraph is an invex set. These results give evidence that invexity can be substituted for convexity in many results in mathematical fuzzy programming involving convex fuzzy mappings. However, the proofs contain some errors. This paper offers correct proofs, by slightly modifying the proof given by Noor, and we also present some new results for preinvex fuzzy mappings, strictly preinvex fuzzy mappings, and quasi-preinvex fuzzy mappings.

In Section 2, some preparatory expositions on the fuzzy numbers and on preinvex fuzzy mappings are given. In Section 3, we introduce the concepts of quasipreinvexity and quasipreincavity for fuzzy mappings based on the fact that $\sup\{u,v\}$ and $\inf\{u,v\}$ exist in $\mathcal{F}_0$ for any pair $\{u,v\} \subset \mathcal{F}_0$, and present the main results of this paper. Section 4 gives some applications of preinvex fuzzy mappings and preincave fuzzy mappings.

2. PRELIMINARIES

Let R^1 denote the set of all real numbers. In this paper, a fuzzy number will be a fuzzy set $u : R^1 \to [0,1]$ which is normal, fuzzy convex, upper semicontinuous, and has compact support. The family of all fuzzy numbers will be denoted by $\mathcal{F}_0$. Since each $r \in R^1$ can be considered as

0898-1221/99/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX
PII: S0898-1221(99)00044-9

a fuzzy number $\tilde{r}$ defined by

$$\tilde{r}(x) = \begin{cases} 1, & \text{if } x = r, \\ 0, & \text{if } x \neq r, \end{cases}$$

R^1 can be embedded in $\mathcal{F}_0$.

As is known [3], the α-level set of a fuzzy number $u \in \mathcal{F}_0$ is a closed and bounded interval

$$[a(\alpha), b(\alpha)] = [u]_\alpha = \begin{cases} \left\{x \in R^1 \mid u(x) \geq \alpha\right\}, & \text{if } 0 < \alpha < 1, \\ \text{cl(supp } u), & \text{if } \alpha = 0, \end{cases}$$

where cl(supp u) denotes the closure of the support of u.

In the analysis of fuzzy numbers, the use of α-level sets of a fuzzy set is simpler than the use of the membership function of a fuzzy set. We recall the following.

LEMMA 2.1. *(See [3, Section 6.1] or [4, Lemma 2.2].) Let $\{[a(\alpha), b(\alpha)] \mid 0 \leq \alpha \leq 1\}$ be a given family of nonempty closed and bounded intervals of R^1 such that*

(1) *if $0 \leq \alpha_1 \leq \alpha_2$ implies that*

$$[a(\alpha_2), b(\alpha_2)] \subseteq [a(\alpha_1), b(\alpha_1)],$$

(2) *for any nondecreasing sequence $\{\alpha_k\}$ in $[0,1]$ converging to α,*

$$[a(\alpha), b(\alpha)] = \bigcap_{k=1}^{\infty} [a(\alpha_k), b(\alpha_k)].$$

Then the family $[a(\alpha), b(\alpha)]$ represents the α-level sets of a fuzzy number $u \in \mathcal{F}_0$.

Conversely, if $[a(\alpha), b(\alpha)]$, $0 \leq \alpha \leq 1$, are the α-level sets of a fuzzy number $u \in \mathcal{F}_0$, then Conditions (1) and (2) are satisfied.

It can easily be verified that the intersection of an arbitrary collection of convex sets is convex. Let $\mathcal{I}$ denote the collection of all nonempty closed and bounded intervals of R^1. Then by Proposition 2.4.5 in [3], we have the following.

LEMMA 2.2. *Let $\{I_n\} \subset \mathcal{I}$ satisfy*

$$\cdots \subseteq I_n \subseteq \cdots \subseteq I_2 \subseteq I_1.$$

Then $\tilde{I} = \cap_{n=1}^{\infty} I_n \in \mathcal{I}$ and

$$d_H\left(I_n, \tilde{I}\right) \to 0, \qquad \text{as } n \to \infty,$$

where d is the Hausdorff metric defined in $\mathcal{I}$.

From Lemma 2.1 we see that a fuzzy number $u : R^1 \to [0,1]$ is determined by the endpoints of the intervals $[u]_\alpha$. Thus, we can identify a fuzzy number u with the parameterized triples

$$\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\},$$

where $a(\alpha)$ and $b(\alpha)$ denote the left- and right-hand endpoints of $[u]_\alpha$, respectively. Suppose that $u, v \in \mathcal{F}_0$ are fuzzy numbers represented by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ and $\{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$, respectively. Define a partial ordering $\preceq$ in $\mathcal{F}_0$ by

$$u \preceq v, \text{ if and only if } a(\alpha) \leq c(\alpha), \text{ and } b(\alpha) \leq d(\alpha), \qquad \text{for all } \alpha \in [0,1].$$

We say that $u \prec v$, if $u \preceq v$ and there exists $\alpha_0 \in [0,1]$ such that

$$a(\alpha_0) < c(\alpha_0) \quad \text{or} \quad b(\alpha_0) < d(\alpha_0).$$

We see that $u = v$, if $u \preceq v$ and $v \preceq u$. It is often convenient to write $v \succeq u$ (respectively, $v \succ u$) in place of $u \preceq v$ (respectively, $u \prec v$).

A subset S^* of $\mathcal{F}_0$ is said to be bounded above if there exists a fuzzy number $u \in \mathcal{F}_0$, called an upper bound of S^*, such that $v \preceq u$ for every $v \in S^*$. Further, a fuzzy number $u_0 \in \mathcal{F}_0$ is called the least upper bound (sup in short) for S^* if (i) u_0 is an upper bound of S^*, and (ii) $u_0 \preceq u$ for every upper bound u of S^*. A lower bound and the greatest lower bound (inf in short) are defined similarly.

For fuzzy numbers $u, v \in \mathcal{F}_0$ represented by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\}$ and $\{(c(\alpha), d(\alpha), \alpha) \mid 0 \le \alpha \le 1\}$, respectively, and each real number r, we define the addition $u + v$ and 'scalar' multiplication ru as follows:

$$\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\} + \{(c(\alpha), d(\alpha), \alpha) \mid 0 \le \alpha \le 1\} \\ = \{(a(\alpha) + c(\alpha), b(\alpha) + d(\alpha), \alpha) \mid 0 \le \alpha \le 1\},$$

$$ru = \{(ra(\alpha), rb(\alpha), \alpha) \mid 0 \le \alpha \le 1\}.$$

It is known that the addition and the nonnegative scalar multiplication on $\mathcal{F}_0$ defined by the above two equations are equivalent to those derived from the usual extension principle. Furthermore, addition and nonnegative scalar multiplication preserve the order on $\mathcal{F}_0$, and $\mathcal{F}_0$ is closed under these operations. It should be noted that for $u \in \mathcal{F}_0$, ru is not a fuzzy number for $r < 0$. The family of parametric representations of members of $\mathcal{F}_0$ and the parametric representations of their negative scalar multiplications form subsets of the vector space

$$\mathcal{V} = \big\{\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\} \mid a : [0,1] \to R^1 \text{ and } b : [0,1] \to R^1 \text{ are bounded functions}\big\}$$

with addition and scalar multiplication defined levelwise.

Let H denote a vector space over R^1. Let K be a nonempty subset of H and let $\eta : K \times K \to H$ be a mapping. Noor [1] introduced the concepts of preinvex fuzzy mappings and invex sets. The definitions of Noor are somewhat inconsistent. We propose the correct notions by Definitions 2.1 and 2.2.

DEFINITION 2.1. *Let $x \in K$. K is said to be invex at x with respect to (w.r.t. in short) η, if for each $y \in K$, $x + \lambda\eta(y, x) \in K$, $0 \le \lambda \le 1$.*

K is said to be an invex set w.r.t. η, if K is invex at each $x \in K$ w.r.t. to η; and a fuzzy invex set if $H = \mathcal{V}$ and $K \subseteq \mathcal{F}_0$.

It can easily be seen that every convex set $C \subseteq H$ is an invex set w.r.t. $\eta(y, x) = y + (-1)x$.

DEFINITION 2.2. *Let K be an invex set w.r.t. η. A fuzzy mapping $F : K \to \mathcal{F}_0$ is said to be preinvex on K (w.r.t. η), if*

$$F(x + \lambda\eta(y, x)) \preceq (1 - \lambda)F(x) + \lambda F(y),$$

for $\lambda \in (0, 1)$ and $x, y \in K$; and strictly preinvex if strict inequality holds for $x \ne y$.

$F : K \to \mathcal{F}_0$ is said to be preincave on K, if

$$F(x + \lambda\eta(y, x)) \succeq (1 - \lambda)F(x) + \lambda F(y),$$

for $\lambda \in (0, 1)$ and $x, y \in K$; and strictly preincave if strict inequality holds for $x \ne y$.

3. MAIN RESULTS

In this section, we propose the concepts of quasiconvexity and quasiconcavity for fuzzy mappings, and then present some basic properties for these fuzzy mappings. We also give two characterization theorems for preinvex fuzzy mappings, and present some new results for preinvex fuzzy mappings, strictly preinvex fuzzy mappings, and quasi-preinvex fuzzy mappings.

The following two lemmas follow immediately from Lemmas 2.1 and 2.2.

LEMMA 3.1. *Let $u, v \in \mathcal{F}_0$ and $[u]_\alpha = [a_1(\alpha), b_1(\alpha)]$, $[v]_\alpha = [a_2(\alpha), b_2(\alpha)]$ for $\alpha \in [0,1]$. Denote by*

$$\max\{a_1(\alpha), a_2(\alpha)\} = a(\alpha) \quad \text{and} \quad \max\{b_1(\alpha), b_2(\alpha)\} = b(\alpha), \tag{3.1}$$

for all $\alpha \in [0,1]$. Then the family $[a(\alpha), b(\alpha)]$ represents the α-level sets of a fuzzy number $w \in \mathcal{F}_0$. Moreover, $w = \sup\{u, v\}$.

LEMMA 3.2. *Let $u, v \in \mathcal{F}_0$ and $[u]_\alpha = [a_1(\alpha), b_1(\alpha)]$, $[v]_\alpha = [a_2(\alpha), b_2(\alpha)]$ for $\alpha \in [0,1]$. Denote by*

$$\min\{a_1(\alpha), a_2(\alpha)\} = c(\alpha) \quad \text{and} \quad \min\{b_1(\alpha), b_2(\alpha)\} = d(\alpha), \tag{3.2}$$

for all $\alpha \in [0,1]$. Then the family $[c(\alpha), d(\alpha)]$ represents the α-level sets of a fuzzy number $z \in \mathcal{F}_0$. Moreover, $z = \inf\{u, v\}$.

Lemmas 3.1 and 3.2 allow us to define quasiconvexity and quasiconcavity for fuzzy mappings on a nonempty invex set. In what follows, let H denote a vector space over R^1, $K \subseteq H$ a nonempty invex set w.r.t. a mapping $\eta : K \times K \to H$.

DEFINITION 3.1. *A fuzzy mapping $F : K \to \mathcal{F}_0$ is said to be quasi-preinvex on the invex set K, if*

$$F(x + \lambda\eta(y,x)) \preceq \sup\{F(x), F(y)\},$$

for $\lambda \in (0,1)$ and $x, y \in K$; and strictly quasi-preinvex if strict inequality holds for $F(x) \neq F(y)$.

$F : K \to \mathcal{F}_0$ is said to be quasi-preincave, if

$$F(x + \lambda\eta(y,x)) \succeq \inf\{F(x), F(y)\}$$

for $\lambda \in (0,1)$ and $x, y \in K$; and strictly quasi-preincave, if strict inequality holds for $F(x) \neq F(y)$.

Let $0 \leq \lambda \leq 1$ and $u, v \in \mathcal{F}_0$. In view of (3.1),(3.2), and the definitions of addition and scalar multiplication on $\mathcal{F}_0$, we see that

$$\inf\{u, v\} \preceq \lambda u + (1-\lambda)v \preceq \sup\{u, v\}. \tag{3.3}$$

This observation leads to the following.

LEMMA 3.3. *Let $F : K \to \mathcal{F}_0$ be a preinvex (respectively, preincave) fuzzy mapping, then F is quasi-preinvex (respectively, quasi-preincave) on K.*

The following two results characterize preinvex fuzzy mappings.

THEOREM 3.1. *Let K be an invex set w.r.t. η. A fuzzy mapping $F : K \to \mathcal{F}_0$ is preinvex on K if and only if for all $x, y \in K$, $\lambda \in R^1$, and all $u, v \in \mathcal{F}_0$ such that $F(x) \prec u$, $F(y) \prec v$, $0 < \lambda < 1$,*

$$F(x + \lambda\eta(y,x)) \prec (1-\lambda)u + \lambda v. \tag{3.4}$$

PROOF. Let $F : K \to \mathcal{F}_0$ be preinvex w.r.t. η, and let $F(x) \prec u$, $F(y) \prec v$, $0 < \lambda < 1$. From the definitions of addition and nonnegative scalar multiplication on $\mathcal{F}_0$, we have

$$\begin{aligned} F(x + \lambda\eta(y,x)) &\preceq (1-\lambda)F(x) + \lambda F(y) \\ &\prec (1-\lambda)u + \lambda v. \end{aligned}$$

Conversely, suppose (3.4) holds. Let $x, y \in K$ and $[F(x)]_\alpha = [a_1(\alpha), b_1(\alpha)]$, $[F(y)]_\alpha = [a_2(\alpha), b_2(\alpha)]$ for $\alpha \in [0,1]$. For any $\delta > 0$, let $u(\delta)$ and $v(\delta)$ be the fuzzy numbers whose α-level sets are given by $[u(\delta)]_\alpha = [a_1(\alpha) + \delta, b_1(\alpha) + \delta]$ and $[v(\delta)]_\alpha = [a_2(\alpha) + \delta, b_2(\alpha) + \delta]$, respectively, then $F(x) \prec u(\delta)$, $F(y) \prec v(\delta)$. So, by (3.4), for $0 < \lambda < 1$,

$$F(x + \lambda\eta(y,x)) \prec (1-\lambda)u(\delta) + \lambda v(\delta),$$

or in terms of the left- and right-hand endpoints of α-level sets

$$a(\alpha) < (1-\lambda)a_1(\alpha) + \lambda a_2(\alpha) + \delta$$

and

$$b(\alpha) < (1-\lambda)b_1(\alpha) + \lambda b_2(\alpha) + \delta,$$

where $[a(\alpha), b(\alpha)] = [F(x + \lambda\eta(y,x))]_\alpha$. Since $\delta > 0$ can be arbitrarily small, it follows that

$$a(\alpha) \le (1-\lambda)a_1(\alpha) + \lambda a_2(\alpha) \quad \text{and} \quad b(\alpha) \le (1-\lambda)b_1(\alpha) + \lambda b_2(\alpha).$$

So, we have

$$F(x + \lambda\eta(y,x)) \preceq (1-\lambda)F(x) + \lambda F(y),$$

for $\lambda \in (0,1)$, and hence, F is preinvex on K. This completes the proof.

THEOREM 3.2. *Let K be an invex set w.r.t. η. A fuzzy mapping $F : K \to \mathcal{F}_0$ is preinvex on K if and only if the set*

$$G_F = \{(x,u) : x \in K, u \in \mathcal{F}_0, F(x) \prec u\}$$

is an invex set w.r.t. the mapping $\eta' : G_F \times G_F \to H \times \mathcal{V}$ with

$$\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u),$$

for (x,u), $(y,v) \in G_F$.

PROOF. Let $F : K \to \mathcal{F}_0$ be preinvex on K. Let $(x,u) \in G_F$ and $(y,v) \in G_F$, i.e., $F(x) \prec u$ and $F(y) \prec v$. Since F is preinvex on K, we have

$$F(x + \lambda\eta(y,x)) \preceq (1-\lambda)F(x) + \lambda F(y) \prec (1-\lambda)u + \lambda v,$$

for all $\lambda \in (0,1)$. It follows that

$$(x + \lambda\eta(y,x), (1-\lambda)u + \lambda v) \in G_F.$$

Thus, all points of the form

$$(x,u) + \lambda(\eta(y,x), v + (-1)u) \in G_F, \qquad \lambda \in [0,1]$$

belong to G_F. Hence, G_F is an invex set w.r.t. $\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u)$.

Conversely, let G_F be an invex set w.r.t. the mapping $\eta' : \text{epi}(F) \times \text{epi}(F) \to H \times \mathcal{V}$ with $\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u)$. Let $x, y \in K$ and $u, v \in \mathcal{F}_0$ such that $F(x) \prec u$, $F(y) \prec v$. Then $(x,u) \in G_F$ and $(y,v) \in G_F$. Now, for $\lambda \in [0,1]$, since G_F is an invex set w.r.t. $\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u)$, $0 < \lambda < 1$, we must have

$$(x,u) + \lambda\eta'((y,v),(x,u)) \in G_F.$$

It follows that

$$(x + \lambda\eta(y,x), (1-\lambda)u + \lambda v) \in G_F.$$

So we have

$$F(x + \lambda\eta(y,x)) \prec (1-\lambda)u + \lambda v.$$

Then, by Theorem 3.1, $F : K \to \mathcal{F}_0$ is preinvex on K. This completes the proof.

Noor proved, by Theorem 3.4 in [1], that a fuzzy mapping $F : K \to \mathcal{F}_0$ is preinvex w.r.t. η if and only if $\text{epi}(F) = \{(x,u) : x \in K, u \in \mathcal{F}_0, F(x) \preceq u\}$ is an invex set w.r.t. η. This statement contains an error, and we give the correct statement by the following.

THEOREM 3.3. *Let K be an invex set w.r.t. η. A fuzzy mapping $F : K \to \mathcal{F}_0$ is preinvex on K if and only if*

$$\mathrm{epi}(F) = \{(x,u) : x \in K, u \in \mathcal{F}_0, F(x) \preceq u\}$$

is an invex set w.r.t. the mapping $\eta' : \mathrm{epi}(F) \times \mathrm{epi}(F) \to H \times \mathcal{V}$ with

$$\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u),$$

for (x,u), $(y,v) \in \mathrm{epi}(F)$.

PROOF. Let $F : K \to \mathcal{F}_0$ be preinvex on K. Let $(x,u) \in \mathrm{epi}(F)$ and $(y,v) \in \mathrm{epi}(F)$. Since F is preinvex on K,

$$F(x + \lambda\eta(y,x)) \preceq (1-\lambda)F(x) + \lambda F(y) \preceq (1-\lambda)u + \lambda v,$$

for $0 \le \lambda \le 1$. Hence,

$$(x + \lambda\eta(y,x), (1-\lambda)u + \lambda v) \in \mathrm{epi}(F),$$

which implies that

$$(x,u) + \lambda(\eta(y,x), v + (-1)u) \in \mathrm{epi}(F).$$

Thus, $\mathrm{epi}(F)$ is an invex set w.r.t. the mapping $\eta' : \mathrm{epi}(F) \times \mathrm{epi}(F) \to H \times \mathcal{V}$ with $\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u)$, for (x,u), $(y,v) \in \mathrm{epi}(F)$.

Conversely, let $\mathrm{epi}(F)$ be an invex set w.r.t. the mapping $\eta' : \mathrm{epi}(F) \times \mathrm{epi}(F) \to H \times \mathcal{V}$ with $\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u)$, for (x,u), $(y,v) \in \mathrm{epi}(F)$. Since $(x, F(x))$, $(y, F(y)) \in \mathrm{epi}(F)$, we have for $0 \le \lambda \le 1$,

$$(x, F(x)) + \lambda\eta'((y, F(y)), (x, F(x)) \in \mathrm{epi}(F),$$

which implies that

$$(x + \lambda\eta(y,x), (1-\lambda)F(x) + \lambda F(y)) \in \mathrm{epi}(F).$$

So we have

$$F(x + \lambda\eta(y,x)) \preceq (1-\lambda)F(x) + \lambda F(y).$$

Hence, F is preinvex on K and this completes the proof.

THEOREM 3.4. *Let K be an invex set w.r.t. η. A fuzzy mapping $F : K \to \mathcal{F}_0$ is quasi-preinvex on K if and only if for every $u \in \mathcal{F}_0$, the set*

$$\Lambda_u = \{x \in K : F(x) \preceq u\}$$

is an invex set w.r.t. η.

PROOF. Let $F : K \to \mathcal{F}_0$ be quasi-preinvex on K, and let $u \in \mathcal{F}_0$. If F_u is an empty set, then it is obvious an invex set w.r.t. η. Assume that x, $y \in \Lambda_u$, i.e., $F(x) \preceq u$ and $F(y) \preceq u$. Then u is an upper bound of $\{F(x), F(y)\}$, and so

$$\sup\{F(x), F(y)\} \preceq u.$$

Since $F : K \to \mathcal{F}_0$ is quasi-preinvex w.r.t. η, from the above inequality, we obtain

$$\begin{aligned} F(x + \lambda\eta(y,x)) &\preceq \sup\{F(x), F(y)\} \\ &\preceq u, \end{aligned}$$

for all $\lambda \in (0,1)$. Thus, all points of the form $x + \lambda\eta(y,x)$, $\lambda \in [0,1]$ belong to Λ_u. Hence, Λ_u is an invex set w.r.t. η.

Conversely, assume that for every $u \in \mathcal{F}_0$, $\Lambda_u = \{x \in K : F(x) \preceq u\}$ is an invex set w.r.t. η. Let x, $y \in K$ and $\tilde{u} = \sup\{F(x), F(y)\}$. Then we have $F(x) \preceq \tilde{u}$ and $F(y) \preceq \tilde{u}$, i.e., $x \in \Lambda_{\tilde{u}}$ and $y \in \Lambda_{\tilde{u}}$. Since $\Lambda_{\tilde{u}}$ is an invex set w.r.t. η, it follows that $x + \lambda\eta(y,x) \in \Lambda_{\tilde{u}}$. Hence,

$$F(x + \lambda\eta(y,x)) \preceq \tilde{u} = \sup\{F(x), F(y)\}.$$

Thus, $F : K \to \mathcal{F}_0$ is quasi-preinvex on K. This completes the proof.

COROLLARY 3.1. *Let K be an invex set w.r.t. η, and let $F : K \to \mathcal{F}_0$ be preinvex on K. Then for every $u \in \mathcal{F}_0$, the set*

$$\Lambda_u = \{x \in K : F(x) \preceq u\}$$

is an invex set w.r.t. η.

PROOF. Let $F : K \to \mathcal{F}_0$ be preinvex K. Then by Lemma 3.3, F is quasi-preinvex on K, and hence, for every $u \in \mathcal{F}_0$, $\Lambda_u = \{x \in K : F(x) \preceq u\}$ is an invex set w.r.t. η by Theorem 3.4.

Corresponding to Theorem 3.4, quasi-preincave fuzzy mappings satisfy the opposite inequalities under similar hypotheses.

COROLLARY 3.2. *Let K be an invex set w.r.t. η. A fuzzy mapping $F : K \to \mathcal{F}_0$ is quasi-preincave on K if and only if for every $u \in \mathcal{F}_0$, the set*

$$\{x \in K : F(x) \succeq u\}$$

is an invex set w.r.t. η.

PROOF. The idea of the proof of this corollary is quite similar to that of Theorem 3.4.

Since addition and nonnegative scalar multiplication on $\mathcal{F}_0$ is closed, it can be seen easily that $\mathcal{F}_0$ is a convex subset of the vector space $\mathcal{V}$. This enables us to speak of convex fuzzy mappings on $\mathcal{F}_0$.

THEOREM 3.5. *Let K be an invex set w.r.t. η, and let $F : K \to \mathcal{F}_0$ be a preinvex fuzzy mapping on K. If $G : \mathcal{F}_0 \to \mathcal{F}_0$ is a nondecreasing convex fuzzy mapping, then the mapping $x \longmapsto G(F(x))$ is preinvex on K.*

PROOF. Let x, $y \in K$, and $\lambda \in [0, 1]$. Since $F : K \to \mathcal{F}_0$ is preinvex on K, we have

$$F(x + \lambda\eta(x, y)) \preceq (1 - \lambda)F(x) + \lambda F(y).$$

Since $G : \mathcal{F}_0 \to \mathcal{F}_0$ is nondecreasing and convex, it follows that

$$\begin{aligned} G(F(x + \lambda\eta(x, y))) &\preceq G((1 - \lambda)F(x) + \lambda F(y)) \\ &\preceq (1 - \lambda)G(F(x)) + \lambda G(F(y)), \end{aligned}$$

and hence, $x \longmapsto G(F(x))$ is preinvex on K. This completes the proof.

THEOREM 3.6. *Let K be an invex set w.r.t. η, and let $F : K \to \mathcal{F}_0$ be a preincave fuzzy mapping on K. If $G : \mathcal{F}_0 \to \mathcal{F}_0$ is a nondecreasing concave fuzzy mapping, then the mapping $x \longmapsto G(F(x))$ is preincave on K.*

PROOF. The idea of the proof is similar as that of Theorem 3.5.

4. APPLICATIONS

We now discuss some applications of preinvex fuzzy mappings to optimization theory.

THEOREM 4.1. *Let K be an invex set w.r.t. η. Suppose that F is a fuzzy mapping defined on K such that $\inf\{F(x) : x \in K\}$ exists in $\mathcal{F}_0$. Let $\mu = \inf\ \{F(x) : x \in K\}$.*

(1) *If $F : K \to \mathcal{F}_0$ is preinvex on K, then the set*

$$\Omega = \{x \in K : F(x) = \mu\}$$

is an invex set w.r.t. η.

(2) *If F is strictly preinvex, then Ω is singleton or empty. That is, if F is strictly preinvex, then F has at most one global minimum point.*

PROOF. To prove part (1), let $F : K \to \mathcal{F}_0$ be preinvex on K. If Ω is an empty set, then it is obvious an invex set w.r.t. η. Assume that x, $y \in \Omega$, i.e., $F(x) = F(y) = \mu$. Since F is a preinvex fuzzy mapping on K,

$$\begin{aligned} F(x + \lambda\eta(y,x)) &\preceq (1-\lambda)F(x) + \lambda F(y) \\ &= (1-\lambda)\mu + \lambda\mu \\ &= \mu, \end{aligned}$$

for all $\lambda \in (0,1)$. Thus, all points of the form $x + \lambda\eta(y,x)$, $\lambda \in [0,1]$ belong to Ω. Hence, Ω is an invex set w.r.t. η.

For the second part, the proof will be by contradiction. Assume that there exist distinct points x, $y \in K$ such that $F(x) = F(y) = \mu$. Since K is an invex set w.r.t. η, then for $\lambda \in (0,1)$, $x + \lambda\eta(y,x) \in K$. Further, since F is strictly preinvex on K,

$$\begin{aligned} F(x + \lambda\eta(y,x)) &\prec (1-\lambda)F(x) + \lambda F(y) \\ &= (1-\lambda)\mu + \lambda\mu \\ &= \mu. \end{aligned}$$

This contradicts that $\mu = \inf \{F(x) : x \in K\}$, and hence, the result follows.

COROLLARY 4.1. *Let K be an invex set w.r.t. η. Suppose that F is a fuzzy mapping defined on K such that* $\sup\{F(x) : x \in K\}$ *exists in $\mathcal{F}_0$. Let $\nu = \sup\{F(x) : x \in K\}$.*

(1) *If $F : K \to \mathcal{F}_0$ is preincave on K, then the set*

$$\Gamma = \{x \in K : F(x) = \nu\}$$

is an invex set w.r.t. η.

(2) *If F is strictly preincave, then Γ is singleton or empty. That is, if F is strictly preincave, then F has at most one global maximum point.*

PROOF. The idea of the proof is similar to that of Theorem 4.1.

It is known [5] that every nonempty set $S^* \subseteq \mathcal{F}_0$ which is bounded above (respectively, bounded below) has a least upper (respectively, greatest lower) bound. Now, we introduce the following device for constructing invex fuzzy mappings.

THEOREM 4.2. *Let $A = K \times S^* \subseteq H \times \mathcal{V}$, where K is a nonempty subset of H and $S^* \subseteq \mathcal{F}_0$ is nonempty and bounded below, be an invex set w.r.t. a mapping $\eta' : A \times A \to H \times \mathcal{V}$ satisfying*

$$\eta'((y,v),(x,u)) = (\eta(y,x), v + (-1)u),$$

for (x,u), $(y,v) \in A$, $x \in K$, $y \in K$, u, $v \in \mathcal{F}_0$, and let

$$F(x) = \inf\{u : (x,u) \in K \times S^*\}. \tag{4.1}$$

Then K is an invex set w.r.t. η, and $F : K \to \mathcal{F}_0$ is a preinvex fuzzy mapping on K.

PROOF. Since $S^* \subseteq \mathcal{F}_0$ is nonempty and bounded below, from (4.1), we see that $F(x)$ is a fuzzy number for each $x \in K$. Thus, F is a fuzzy mapping on K. By the invexity of the set $K \times S^*$, it is to check that K is an invex set w.r.t. η. Now, it suffices to show that the fuzzy mapping $F : K \to \mathcal{F}_0$ is preinvex on K. To see this, let x, $y \in K$. Since A is an invex set w.r.t. η', it is easily verified that if (x,u), $(y,v) \in K \times S^*$, then for each $\lambda \in (0,1)$,

$$(x + \lambda\eta(y,x), (1-\lambda)u + \lambda v) \in K \times S^*. \tag{4.2}$$

In view of (4.1) and (4.2), we obtain

$$F(x + \lambda\eta(y,x)) \preceq (1-\lambda)F(x) + \lambda F(y),$$

for each $\lambda \in (0,1)$. Hence, the fuzzy mapping $F : K \to \mathcal{F}_0$ is preinvex on K.

Now let H be a normed vector space over R^1. Noor [1] proved that any local minimum of a preinvex fuzzy mapping $F : K \to \mathcal{F}_0$, where $K \subseteq H$ being an invex set, is a global minimum of F on K. But his proof contains an error. We offer a correct proof by slightly modifying Noor's proof.

THEOREM 4.3. *Let K be a nonempty invex set w.r.t. η. Suppose that $F : K \to \mathcal{F}_0$ is a preinvex fuzzy mapping on K such that* $\inf\{F(x) : x \in K\}$, *say μ, exists in $\mathcal{F}_0$, and that the set*

$$\Omega = \{x \in K : F(x) = \mu\} \neq \emptyset.$$

If $\tilde{x}$ is a local minimum of F, then it is also a global minimum of F on K.

PROOF. Let $\tilde{x}$ be a local minimum of F. If $\tilde{x}$ is not a global minimum of F on K, then $\tilde{x} \notin \Omega$. By the hypothesis $\Omega \neq \emptyset$, let $y \in \Omega$, we must have $F(y) \prec F(\tilde{x})$. Since F is preinvex on K, there exists $\eta : K \times K \to H$ such that for $\lambda \in (0,1)$

$$\begin{aligned} F(\tilde{x} + \lambda\eta(y, \tilde{x})) &\preceq (1 - \lambda)F(\tilde{x}) + \lambda F(y) \\ &\prec F(\tilde{x}) : \text{since } F(y) \prec F(\tilde{x}). \end{aligned}$$

So

$$F(\tilde{x} + \lambda\eta(y, \tilde{x})) \prec F(\tilde{x})$$

for arbitrary small positive number λ, and this contradiction proves the result.

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